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Axially symmetric solutions in general relativity

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Abstract. Two new sets of asymptotically flat and functionally non-related electrovac solutions representing the external field of an isolated mass carrying electric charge, dipole and higher multipole moments are presented and a stationary vacuum metric is generated.

1. Introduction

Long ago, Weyl (1917) obtained a class of axially symmetric electrovac solutions in which the gravitational and electrostatic potentials are functionally related. Another class of solutions without spatial symmetry but with functionally related potentials was found by Majumdar (1947) and by Papapetrou (1947). Bonnor (1966) obtained a solution without functionally related potentials, referring to a massive electric dipole by transforming the Kerr (1963) stationary vacuum metric into an electrostatic space–time. This procedure (henceforth referred to as the parameter change technique) has since been extended to the Tomimatsu and Sato vacuum solutions (1973) for $\delta = 1-4$ by several authors (see e.g. Wang 1974, Das and Banerji 1978 etc) to yield some extremely complicated space–times.

Recently Chandrasekhar (1978) has reformulated the field equations of an axially symmetric stationary vacuum problem in terms of two real functions instead of the complex formalism of Ernst (1968) and will display a derivation of Kerr's solution by a simple method, better than any of the formulations considered so far. Chandrasekhar's work has also provided new stationary solutions generated from the old ones, but these have not yet been looked into in great detail.

Bonnor (1979), using Chandrasekhar's formula (1978) has again obtained a new class of functionally non-related electrovac solutions representing the exterior field of a body with mass, electrostatic monopole and higher multipoles from the Kerr metric. This procedure will henceforth be referred to as the C–B technique.

Kinnersley and Chitre (1978a, b) have given a new stationary axially symmetric vacuum metric different from the metric of Tomimatsu and Sato (1973) for $\delta = 2$. In this paper we have utilised Chandrasekhar's technique of obtaining a new stationary vacuum solution from the solution of Kinnersley and Chitre (1978a) and also obtained two new sets of functionally non-related axially symmetric electrovac solutions by the C–B technique. All the electrovac solutions generated are asymptotically flat.

In § 2 Chandrasekhar's prescription is briefly outlined and a simple method of obtaining real solutions of Chandrasekhar's equations from the known solutions of Ernst's complex formalism is discussed. A new electrovac solution is generated in § 3

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from the stationary vacuum solution of Kinnersley and Chitre (1978a) and compared with the corresponding electrovac solution obtained by the parameter change technique. In § 4 a new electrovac solution is obtained from the T–S, $\delta=2$ stationary vacuum solution by the C–B technique and compared accordingly with the known electrovac solution obtained by the parameter change technique. New stationary vacuum metric is generated from the solution of Kinnersley and Chitre (1978a) by Chandrasekhar's prescription (1978) in § 5. The paper ends with a discussion of using different techniques for the same generating function.

2. Chandrasekhar's technique

Chandrasekhar (1978) chose the line element in a more general form and derived the Kerr metric directly, verifiable at all stages. Secondly he suggests a method for the generation of explicit classes of exact solutions and gives an example of the above class. We are concerned particulary with the new method here.

Chandrasekhar's line element is written as:

$$ds^2=-\exp(2\nu)\,dt^2+\exp(2\psi)(d\varphi-w\,dt)^2+\exp(2u_2)(dx^2)^2+\exp(2u_3)(dx^3)^2 \quad (2.1)$$

where φ denotes the azimuthal angles (in the equatorial plane) and $x^2(=r)$ and $x^3(=\theta)$ are the two remaining spatial coordinates. In equations (2.1) v, ψ, ω, u_2 and u_3 are by the assumptions of stationary and axisymmetry, functions of x^2 and x^3 only. With suitable transformation Chandrasekhar finally writes the line element (2.1) in the following form

$$ds^2=(\Delta\delta)^{1/2}\{\chi(dt^2)+\chi^{-1}(d\phi-\omega\,dt)^2\}+\Delta^{-1/2}\exp(u_2+u_3)\{(dr^2)+\Delta(d\theta^2)\} \quad (2.2)$$

which reduces the field equations into

$$\tfrac{1}{2}(X+Y)\{(\Delta X_{,r})_{,r}+(\delta X_{,u})_{,u}\}=\Delta(X_{,r})^2+\delta(X_{,u})^2, \quad (2.3)$$

$$\tfrac{1}{2}(X+Y)\{(\Delta Y_{,r})_{,r}+(\delta Y_{,u})_{,u}\}=\Delta(Y_{,r})^2+\delta(Y_{,u})^2. \quad (2.4)$$

Definitions of the variables

$$X=\chi+\omega, \qquad Y=\chi-\omega, \qquad u=\cos\theta, \qquad \Delta^{1/2}=\exp(u_3-u_2)$$
$$\delta=1-u^2, \qquad \chi=\exp(-\psi+\nu) \quad (2.5)$$

where u_2 and u_3 are determined by quadrature. This approach by Chandrasekhar has the advantage that it does not initially assume the cylindrical symmetry of canonical coordinates.

A convenient form of equations (2.3) and (2.4) which enables one to find some solutions from solutions of the Ernst equation is obtained by the transformations:

$$X=(1+F)/(1-F) \qquad Y=(1+G)/(1-G)$$

and

$$\eta=(r-M)/(M^2-a^2)^{1/2} \qquad \Delta=(M^2-a^2)(\eta^2-1) \quad (2.7)$$

where η and u are easily identified as the spatial coordinates x and y in the prolate spheroidal coordinate system.

We obtain equations (2.3) and (2.4) in the forms

$$(1-FG)\{[(x^2-1)F_{,x}]_{,x}+[(1-y^2)F_{,y}]_{,y}\}=-2G[(x^2-1)(F_{,x})^2+(1-y^2)(F_{,y})^2] \tag{2.8}$$

$$(1-FG)\{[(x^2-1)G,x]_{,x}+[(1-y^2)G,y]_{,y}=-2F[(x^2-1)(G,x)^2+(1-y^2)(G,y)^2]. \tag{2.9}$$

Thus once we obtain the solution for F and G of equations (2.8) and (2.9), the metric (2.2) is solved. This is a new class of solution which results from Chandrasekhar's new formalism. Furthermore, it has been shown by Bonnor (1979) that the solution of equations (2.8) and (2.9) entails also the exact solution of axially symmetric static electrovac fields. Equations (2.8) and (2.9) are then of fundamental importance and Chandrasekhar obtained the solution of (2.8) and (2.9) from the Ernst equation by simple inspection. We also require (§§ 3, 4, 5) the solutions of equations (2.8) and (2.9) in a different form from that of Chandrasekhar and therefore give in the following, the mathematical basis for directly obtaining the solution of the above two equations if we know the solution of the Ernst equation in the form

$$\xi=f(x,y,q)+\mathrm{i}q\phi(x,y,q), \qquad (q \text{ is a const}). \tag{2.10}$$

Actually the T–S solutions for $\delta=1-4$ and the Kinnersley and Chitre solution (1978a) are reducible to the form (2.10).

2.1. *Method of obtaining solutions*

Ernst's (1968) celebrated equation is written as:

$$(1-\xi\xi^*)\{[(x^2-1)\xi_x]_{,x}+[(1-y^2)\xi_y]_{,y}\}=-2\xi^*[(x^2-1)\xi_x^2+(1-y^2)\xi_y^2]. \tag{2.11}$$

Separating equation (2.11) into real and imaginary parts when $\xi=f+\mathrm{i}q\phi$ we get

$$\begin{aligned}(1-f^2-q^2\phi^2)\{[(x^2-1)f_x]_{,x}+[(1-y^2)fy],_{,y}\}\\ =-2f[(x^2-1)(f_x^2-q^2\phi_x^2)+(1-y^2)(f_y^2-q^2\phi_y^2)]\\ +4(\mathrm{i}q)^2\phi[(x^2-1)f_x\phi_x+(1-y^2)fy\phi y],\end{aligned} \tag{2.12}$$

$$\begin{aligned}(1-f^2-q^2\phi^2)\{[(x^2-1)\phi_x]_{,x}+[(1-y^2)\phi_y]_{,y}\}\\ =-4f[(x^2-1)f_x\phi_x+(1-y^2)f_y\phi_y]+2\phi[(x^2-1)(f_x^2-q^2\phi_x^2)\\ +(1-y^2)(f_y^2-q2\phi_y^2)].\end{aligned} \tag{2.13}$$

Similarly writing $F=f'+k\phi'$ and $G=f'-k\phi'$ ($k=\text{const}$) equations (2.8) and (2.9) reduce to two independent equations.

$$\begin{aligned}(1-f'^2+k^2\phi'^2)\{[(x^2-1)f'_x]_{,x}+[(1-y^2)f'_y]_{,y}\}\\ =-2f'[(x^2-1)(f_x'^2+k^2\phi_x'^2)+(1-y^2)(f_y'^2+k^2\phi_y'^2)]\\ +4k^2\phi'[(x^2-1)f'_x\phi'_x+(1-y^2)f'_y\phi'_y]\end{aligned} \tag{2.14}$$

$$\begin{aligned}(1-f'^2+k^2\phi'^2)\{[(x^2-1)k\phi'_x]_{,x}+[(1-y^2)k\phi'_y]_{,y}\}\\ =-4kf'[(x^2-1)f'_x\phi'_x+(1-y^2)f'_y\phi'_y]\\ +2k\phi'[(x^2-1)(f_x'^2+k^2\phi_x'^2)+(1-y^2)(f_y'^2+k^2\phi_y'^2)].\end{aligned} \tag{2.15}$$

A comparison shows that the two pairs of equations (2.12) and (2.13) and (2.14) and

(2.15) are related to each other in the same way as that of the two pairs we encountered in the parameter change technique (Das and Banerji 1978, equations (5–6) and (8–9)). Thus we can get f' and ϕ' from f and ϕ respectively by changing all the iq's present in f and ϕ into k.

$$F = f' + q\phi' \qquad \text{and} \qquad G = f' - q\phi'. \tag{2.16}$$

This is found to be true in the case of T–S, $\delta = 1-4$ solutions and also in Kinnersley and Chitre's solution (1978). No intuition or inspection is then necessary to obtain real solutions of F and G from the complex solutions of Ernst's equation.

3. The new electrovac solutions

Here we use Bonnor's (1979) procedure modified whenever necessary. The line element of axially symmetric electrovac space–times may be written as:

$$ds^2 = -e^{\lambda}(du^2 + d\theta^2) - \alpha^{-2}\Delta^2 d\varphi^2 + \alpha^2 dt^2 \tag{3.1}$$

where $x^1 \equiv u$, $x^2 \equiv \theta$, $x^3 \equiv \varphi$ and λ, Δ, α and ϕ (defined later) are functions of u and θ only. The equations to be solved are those of Einstein–Maxwell theory in the absence of matter

$$R_{ik} = 2F_i^a F_{ka} - \tfrac{1}{2} g_{ik} F^{ab} F_{ab} \tag{3.2}$$

where R_{ik} is the Ricci tensor and F_{ik} is the electromagnetic tensor which satisfies Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \tag{3.3a}$$

$$F^{ik}{}_{;k} = 0. \tag{3.3b}$$

In the electrostatic problem all variables are independent of the time $t(\equiv x^4)$ and (3.3a) is satisfied if we take

$$F_{ik} = k_{i;k} - k_{k;i}, \qquad k_i = \delta_i^4 \phi(x^\nu) \qquad (x^\nu = x^1, x^2, x^3)$$

ϕ being the electrostatic potential. The entire solution is determined by two equations; equation (44) of (3.2) and the equation for $i = 4$ of (3.3b) (Bonnor 1979). Taking linear combinations of those and putting

$$X = \alpha + \phi \qquad Y = \alpha - \phi \tag{3.4}$$

we obtain two equations equivalent to the two given by Chandrasekhar.

$$(X + Y)\nabla^2 X = 2\nabla X \nabla X \tag{3.5a}$$

$$(X + Y)\nabla^2 Y = 2\nabla Y \nabla Y. \tag{3.5b}$$

Once X and Y are found, the function λ in (3.1) is obtained up to an additive constant by other field equations (3.2) (Bonnor 1953). Choosing u and θ to be prolate spheroidal coordinates defined by

$$\eta = \cosh u, \qquad u = \cos\theta \tag{3.6}$$

and transforming X and Y into F and G as given in equation (2.6) we recover the two fundamental equations (2.8) and (2.9) which determine uniquely the electrovac metric (3.1) as well as the new stationary metric (2.1). For monopole solutions, suggested transformations (Chandrasekhar 1978) of X and Y as

$$X \to X(1+c'X), \qquad Y \to Y/(1-c'Y) \tag{3.7}$$

are also taken into consideration whenever necessary.

3.1. The solutions

Kinnersley and Chitre (1978a, equation (13)) have recently given a stationary axially symmetric solution of Einsteins equation and have claimed that their solution is new and different from the T–S, $\delta = 2$ solution. In Ernst's notation their solution is given by

$$\xi = \frac{(x^4-1)-2\mathrm{i}\beta xy(x^2+y^2-2)-\beta^2(x^2-y^2)^2}{2x(x^2-1)+2\mathrm{i}\beta y(x^2-y^2)}, \qquad \beta = \text{const.} \tag{3.8}$$

Now, using the technique discussed in § 2, F and G can be written as:

$$F = \frac{(x^4-1)-2\beta xy(x^2+y^2-2)+\beta^2(x^2-y^2)^2}{2x(x^2-1)+2\beta y(x^2-y^2)} \tag{3.9}$$

$$G = \frac{(x^4-1)+\beta^2(x^2-y^2)^2+2\beta xy(x^2+y^2-2)}{2x(x^2-1)-2\beta y(x^2-y^2)}. \tag{3.10}$$

α and ϕ come out to be

$$\alpha = \tfrac{1}{2}(a_2-a_1)(1-FG)/(a_1a_2+a_1^2G+a_2^2F+a_1a_2FG) \tag{3.11a}$$

$$\phi = \frac{1}{2}\frac{(a_1+a_2)+2a_1G+2a_2F+(a_1+a_2)FG}{a_1a_2+a_1^2G+a_2^2F+a_1a_2FG} \tag{3.11b}$$

where

$$c'+1 = a_1 \qquad \text{and} \qquad c'-1 = a_2\,; \tag{3.12}$$

c' is an arbitrary constant in the transformation equation (3.7).

The electrovac solution (3.11a and 3.11b) is obtained by the C–B technique. With the replacement of actual values of F and G it may appear that a certain common constant is present in both the expressions for α and ϕ. This can be removed by the obvious transformation, $t = (\text{const})^{-1}t'$ and $\phi = (\text{const})\ \phi'$. Moreover, ϕ is always present in the field equations (3.2) and (3.3) as the derivatives, so we are at liberty to introduce another arbitrary constant in the expression for ϕ. These two are necessary for making α^2 tend to unity and ϕ tend to zero at spatial infinity. Asymptotic expansions for α and ϕ are as follows:

$$\alpha = 1 - \frac{2(a_1^2+a_2^2)}{(1+\beta^2)a_1a_2}\frac{1}{x} + \mathrm{O}(x^{-2}) + \ldots \tag{3.13a}$$

$$\phi = -\frac{4(a_1+a_2)}{(1+\beta^2)a_1a_2}\frac{1}{x} + \mathrm{O}(x^{-2}) + \ldots. \tag{3.13b}$$

The asymptotic expansions may also be written in terms of spherical coordinates r, θ and φ by means of the transformation,

$$lx = r - m/2, \qquad y = \cos\theta \tag{3.14}$$

where l and m are constants. The above new solution is asymptotically flat and refers to a three-parameter solution (c, l and β) exterior to a charged massive body with monopole moment and higher multipoles; the charge-to-mass ratio comes out to be

$$e/m = 2c'/(c'^2+1). \tag{3.15}$$

It is interesting to note that when $c'=0$, the monopole term in the expansion of ϕ vanishes and we obtain the two-parameter dipole solution. The parameter-change technique applied to equation (3.8) also gives an asymptotically flat electrovac solution which is new and which has not been found so far. In the following we therefore present such a solution obtained from equation (3.8).

The parameter-change technique is now well known in the literature (for details see Das and Banerji, 1978) and therefore only a synopsis of the same is given. The result is new and interesting.

If the stationary metric

$$ds^2 = -e^{-u}[e^{2\nu}(d\rho^2+dz^2)+\rho^2\, d\varphi^2]+e^u(dt-\omega\, d\varphi)^2 \tag{3.16}$$

is known, the corresponding electrovac metric

$$ds^2 = e^{2\delta}\, dt^2 - e^{-2\delta}[e^{2\gamma'}(d\rho^2+dz^2)+\rho^2\, d\varphi^2], \tag{3.17}$$

can be solved by changing a certain constant present in the solution of (3.16) with a definite rule discussed by Das and Banerji (1978). The two pairs of equations in prolate spheroidal coordinates corresponding to the stationary and electrovac problems respectively show a formal similarity except for a change of sign as shown in the following.

$$(x^2-1)u_{11}+(1-y^2)u_{22}+2xu_1-2yu_2 = -e^{-2u}[(x^2-1)\phi_1^2+(1-y^2)\phi_2^2] \tag{3.18a}$$

$$(x^2-1)\phi_{11}+(1-y^2)\phi_{22}+2x\phi_1-2y\phi_2 = 2[(x^2-1)u_1\phi_1+(1-y^2)u_2\phi_2] \tag{3.18b}$$

$$(x^2-1)\delta_{11}+(1-y^2)\delta_{22}+2x\delta_1-2y\delta_2 = e^{-2\delta}[(x^2-1)\psi_1^2+(1-y^2)\psi_2^2], \tag{3.19}$$

$$(x^2-1)\psi_{11}+(1-y^2)\psi_{22}+2x\psi_1-2y\psi_2 = 2[(x^2-1)\delta_1\psi_1+(1-y^2)\delta_2\psi_2] \tag{3.20}$$

where u and δ are the metric functions in equations (3.16) and (3.17); ϕ and ψ, the twist and electrostatic potential respectively.

The electrovac solution corresponding to metric (3.17) is derived from Kinnersley and Chitre's stationary solution by the parameter change technique and written below.

$$e^{\delta} = 1-4(D/E), \qquad \psi = 4\beta y(F/E) \tag{3.21}$$

where

$$\begin{aligned} D &= x(x^2-1)\{(x+1)^2(x^2-1)+\beta^2(x^2-y^2)^2\}+2\beta^2y^2(x^2-y^2)(x+1)(x^2-2x+y^2), \\ E &= \{(x+1)^2(x^2-1)+\beta^2(x^2-y^2)^2\}^2-4\beta^2y^2(x+1)^2(x^2-2x+y^2)^2, \\ F &= (x^2-y^2)\{(x+1)^2(x^2-1)+\beta^2(x^2-y^2)^2\}+2x(x^2-1)(x+1)(x^2-2x+y^2). \end{aligned} \tag{3.22}$$

The above stated electrovac solution is a new, functionally non-related asymptotically flat dipole solution. With the transformation (3.14) its asymptotic expansion can be given as (see metric (3.1))

$$
\begin{aligned}
e^{2\alpha} &= 1 - \frac{8l}{1+\beta^2}\frac{1}{r} + \frac{4l\{8l - m(1+\beta^2)\}}{(1+\beta^2)^3}\frac{1}{r^2} + O(r^{-3}) + \ldots \\
\phi &= \frac{4\beta(3+\beta^2)l^2}{(1+\beta^2)^2}\frac{\cos\theta}{r^2} + O(r^{-3}) + \ldots
\end{aligned}
\tag{3.23}
$$

Thus this solution represents a source of mass $4l(1+\beta^2)$ and a dipole moment $4\beta(3+\beta^2)l^2/(1+\beta^2)^2$. When $\beta = 0$, the electrostatic potential vanishes and we get Weyl's static vacuum solution of the Einstein equation for $\delta = 2$ as in the case of the electromagnetic analogue of the T–S, $\delta = 2$ solution. Directional singularity at the poles $x = 1$, $y = \pm 1$ is already present in the Weyl static metric and the directional properties of the metric (3.21) is not much different from the T–S metric studied by Economou (1976) and Diaz (1976).

All the solutions generated either by the C–B technique or the parameter-change technique are functionally non-related and well behaved at spatial infinity i.e. g_{44} tends to unity and the electrostatic potential vanishes. When the same generating solution of Kinnersley and Chitre is taken as input, the output electrovac solution by the C–B method refers to the exterior field of a massive body with electrostatic monopole and higher multipole moment whereas the parameter-change technique entails more complicated fields of a massive body with dipole and higher multipole moments.

4. A second set of new electrovac solutions

The method of obtaining electrovac solution by the C–B procedure or the parameter change technique has been discussed in detail in the preceeding sections, we therefore, only quote the results.

4.1. C–B *method*

The Kerr solution, rediscovered by Ernst is the $\delta = 1$ member of the T–S family. From the $\delta = 2$ solution of the T–S solutions we obtain F and G as described in § 2.

$$
\begin{aligned}
F &= \frac{(p^2x^4 - q^2y^4 - 1) - 2pqxy(x^2 - y^2)}{2px(x^2-1) + 2qy(y^2-1)} \\
G &= \frac{(p^2x^4 - q^2y^4 - 1) + 2pqxy(x^2 - y^2)}{2px(x^2-1) - 2qy(y^2-1)}, \qquad (p^2 - q^2) = 1,
\end{aligned}
\tag{4.1}
$$

and a lengthy but straightforward calculation shows that

$$
\begin{aligned}
\alpha &= \frac{a_2 - a_1}{2}\,\frac{(c^2 - d^2 - a^2 + b^2)}{a_1a_2(c^2 - d^2 + a^2 - b^2) + (a_1^2 + a_2^2)(ac+bd) + (a_1^2 - a_2^2)(bc+ad)} \\
\phi &= \frac{1}{2}\,\frac{(a_1 + a_2)(c^2 - d^2 + a^2 - b^2) + 2(a_1 + a_2)(ac+bd) + 2(a_1 - a_2)(bc+ad)}{a_1a_2(c^2 - d^2 + a^2 - b^2) + (a_1^2 + a_2^2)(ac+bd) + (a_1^2 - a_2^2)(bc+ad)}
\end{aligned}
\tag{4.2}
$$

where

$$a = p^2x^4 - q^2y^4 - 1 \qquad b = 2pqxy(x^2 - y^2)$$
$$c = 2px(x^2 - 1) \qquad d = 2qy(y^2 - 1) \tag{4.3}$$
$$a_1 = c' + 1 \qquad a_2 = c' - 1$$

c' being the constant contained in the transformation (3.7).

The asymptotic expansions of α and ϕ are given in the forms,

$$\alpha = 1 - \frac{2(a_1^2 + a_2^2)}{a_1 a_2} \frac{1}{px} - \frac{4q(a_1^2 - a_2^2)y}{a_1 a_2} \frac{1}{p^2x^2} + \dots \tag{4.4a}$$

$$\phi = -\frac{(a_1 + a_2)(a_1^2 - a_2^2)}{a_1 a_2} \frac{1}{px} + \mathrm{O}(px)^{-2} + \dots \tag{4.4b}$$

where suitable transformations and addition of a constant in the expression for ϕ have been taken into consideration to make α and ϕ well behaved at spatial infinity (see metric (3.1)).

The asymptotic behaviour of (4.2) shows that the solution is asymptotically flat at spatial infinity and corresponds to the three-parameter monopole solution, evident with the transformation (3.14). The charge-to-mass ratio is not different from equation (3.15) i.e. the electrovac solutions obtained from Kinnersley and Chitre's solution or the T–S, $\delta = 2$ solution have the same charge-to-mass ratio.

4.2. *The parameter-change technique*

Wang (1974) obtained an electrovac solution from the T–S, $\delta = 2$ solution incorrectly. Das and Banerji (1978) corrected and utilised this for generating further exact solutions by Kinnersley's transformations. Here we present the electrovac solution as presented by Das and Banerji.

$$\alpha = A/B, \qquad \phi = D/B \tag{4.5}$$

where

$$A = p^4(x^2 - 1)^4 + q^4(1 - y^2)^4 + 2p^2q^2(x^2 - 1)(1 - y^2)[2(x^2 - 1)^2 + 2(1 - y^2)^2 + 3(x^2 - 1)(1 - y^2)]$$

$$B = [p^2(x^2 + 1)(x^2 - 1) + q^2(y^2 + 1)(y^2 - 1) + 2px(x^2 - 1)]^2 - 4q^2y^2[px(x^2 - 1) + (px + 1)(1 - y^2)]^2$$

$$D = -4qy(1 - y^2) - 4p^2q[(x^4y)(1 - y^2) + 2x^2y(x^2 - 1)(x^2 - y^2)] - 4q^3y^5(1 - y^2).$$

The asymptotic expansion of ϕ refers to the dipole solution. Thus this can be regarded as the static field of an electric and magnetic dipole

$$\phi \simeq 8qy/p^2x^2 + \dots \tag{4.6}$$

The solution has asymptotic flatness and in the limit $q = 0$, reduces to the Weyl metric with $\delta' = 2\delta$, where δ' is the Weyl parameter.

In closing this section, one important point deserves attention that the parameter change technique generates solutions corresponding to massive electric/magnetic dipole whereas the C–B technique generates massive monopole solution. However, it is seen that F and G obtained directly from the T–S family ($\delta = 1$–4) do not give a monopole solution unless the transformation suggested by Chandrasekhar (3.7) is used. Bonnor (1979) obtained the monopole electrovac field from the solutions of equations (3.5a and b) without taking F and G directly from Kerr's solution. It is seen that taking

$$F = -px - qy \qquad \text{and} \qquad G = -px + qy \tag{4.7}$$

directly from Kerr's solution according to the rule (2.16) we fail to obtain the monopole term in the expansion of the electrostatic potential ϕ. But the transformation (3.7) applied to (4.7) gives the monopole solution as shown below:

$$\begin{aligned} \alpha &= 1 - \frac{2(1+c'^2)}{1-c'^2}\frac{1}{px} - \frac{2}{p^2x^2} + \frac{4c'}{1-c'^2}\frac{qy}{p^2x^2} + \frac{q^2y^2}{p^2x^2} + \ldots \\ \phi &= 1 - \frac{4c'}{1-c'^2}\frac{1}{px} - \frac{1}{p^2x^2}\left\{2qy\frac{1+3c'^2}{c'(c'^2-1)} + \frac{2(1+c'^2)}{c'^2-1}\right\} + \ldots \end{aligned} \tag{4.8}$$

The ratio of e/m comes out to be

$$2c'/(1+c'^2). \tag{4.9}$$

This solution is in some sense linked with Bonnor's solution (1979).

5. The stationary solution

In § 2 it is shown how two functions χ and ω obtained from X and Y generate a new stationary solution. We obtain below new stationary vacuum fields from the solution of Kinnersley and Chitre (1978).

F and G determined from Kinnersley and Chitre, listed in equations (4.1) give X and Y. Again,

$$\chi = \tfrac{1}{2}(X+Y), \qquad \omega = \tfrac{1}{2}(X-Y). \tag{5.1}$$

Transformation (3.7) applied to X and Y gives a new X' and Y', and χ and ω derived from X' and Y' reduce to the same expressions as for α and ϕ listed in (3.11a and b). We therefore refrain from writing out full expressions for χ and ω to avoid repetition. From the asymptotic expansion χ and ω it is seen that $\chi \to 1$ and $\omega \to 0$, at spatial infinity according to equations (3.13a and b). The new stationary solution derived from the Kerr solution by Chandrasekhar without the transformation (3.7) also shows that asymptotically $\chi \to 1$ and $\omega \to 0$, but ω goes more rapidly to zero than the ω in (5.1). Chandrasekhar's reformulation of the stationary axially symmetric field equations provided a beautiful technique for obtaining a new class of monopole electrovac fields but it is not clear at this point how successful his method is for obtaining the physically realistic stationary vacuum metric. This point requires a thorough investigation (Sloane 1978).

6. Conclusion

The two methods, the C–B and the parameter-change technique applied to the same generating function give rise to functionally non-related, asymptotically flat electrovac solutions representing a massive electrostatic monopole with higher multipole fields in the former and a massive dipole and higher multipole in the latter. The parameter-change technique gives more complicated solutions than the C–B method. Solutions derived from the C–B technique are simple and physically realistic. Asymptotically their behaviour does not have the same relevance to nature. For both the methods to be effective we must have exact solutions of the axially symmetric stationary metric to hand and certain constants present in the stationary solution have to be changed at a particular stage of the derivation. Thus it is expected that the C–B method and parameter-change technique are connected to each other by some sort of simple transformation which maps the monopole solution to the dipole one and vice-versa. This problem may be pursued further by the interested reader.

No independent functionally non-related solution (i.e. without transforming the stationary metric to the electrovac one by some method) of equations (3.5*a* and *b*) exists in the literature as yet, so it was not possible to obtain stationary solutions from the independent electrovac fields by reverse transformation of any kind.

It can be seen that any new formulation of the field equations is always fruitful and therefore welcome. Chandrasekhar's recent reformulation is no doubt promising and we, in this paper, have shown clearly that this leads to physically interesting and well behaved electrovac solutions of the Binstein–Maxwell equations, although no definite remark can be made at this moment about this attempt to generate new exact solutions, in particular, the stationary one. Xanthopoulos (1979) has succeeded to some extent in exploring the possibilities.

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